**Ejercicio del Cap. 8 del Curso de TCC de Javier García**
(por Antonio Gros)

Determinar el valor esperado de $\phi_a\phi_b\phi_c\phi_d$ o sea $\langle\phi_a\phi_b\phi_c\phi_d\rangle$

$Z(J) = \int \mathcal{D}\phi e^{-S[\phi]+\phi^T J} = \exp[J^T A^{-1} J]\frac{(\sqrt{2\pi})^n}{m^n\sqrt{\det A}}$
$\langle\phi_a\phi_b\phi_c\phi_d\rangle = \frac{1}{Z(0)}\left[\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\frac{\partial}{\partial J_c}\frac{\partial}{\partial J_d}Z(j)\right]_{J=0} = \frac{\frac{(\sqrt{2\pi})^n}{m^n\sqrt{\det A}}}{\frac{(\sqrt{2\pi})^n}{m^n\sqrt{\det A}}}\left\{\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\frac{\partial}{\partial J_c}\frac{\partial}{\partial J_d}\exp\left[\frac{1}{2m^2}J^T A^{-1} J\right]\right\}$
así que $\langle\phi_a\phi_b\phi_c\phi_d\rangle = \left\{\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\frac{\partial}{\partial J_c}\frac{\partial}{\partial J_d}\exp\left[\frac{1}{2m^2}J^T A^{-1} J\right]\right\}$
Haciendo
$a_{ij} = \frac{1}{2m^2}A^{-1}_{ij}$ nos quedara $(\frac{1}{2m^2}J^T A^{-1} J) = \sum_{ij} a_{ij}J^i J^j = a_{ij}J^i J^j$que inclusve podemos expresar como
$(\frac{1}{2m^2}J^T A^{-1} J) = a_{ij}x^i x^j$ denotando las $J^i$ como $x^i$
Calculemos la derivadas sucesivas de $\exp\left[\frac{1}{2m^2}J^T A^{-1} J\right] = \exp(a_{ij}x^i x^j)$
Respecto a $d$

$\partial_d \exp(a_{ij}x^i x^j) = \exp(a_{ij}x^i x^j)\partial_d(a_{ij}x^i x^j) = \exp(a_{ij}x^i x^j)(a_{ij}x^j\partial_d x^i + a_{ij}x^i\partial_d x^j) = \exp(a_{ij}x^i x^j)(a_{ij}x^j\delta^i_d + a_{ij}x^i\delta^j_d)$

$$\partial_d \exp(a_{ij}x^i x^j) = [\exp(a_{ij}x^i x^j)](a_{dj}x^j + a_{id}x^i) \tag{1}$$

Respecto a $c$
$\partial_c\left[\exp(a_{ij}x^i x^j)\right](a_{dj}x^j + a_{id}x^i) = [\partial_c \exp(a_{ij}x^i x^j)](a_{dj}x^j + a_{id}x^i) + \left[\exp(a_{ij}x^i x^j)\right]\partial_c(a_{dj}x^j + a_{id}x^i)$
La primera derivada ya la tenemos en la expresión recuadrada sin más que remplazar $d$ por $c$
$\partial_c\left[\exp(a_{ij}x^i x^j)\right](a_{dj}x^j + a_{id}x^i) = \left[\exp(a_{ij}x^i x^j)\right](a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + \left[\exp(a_{ij}x^i x^j)\right]\partial_c(a_{dj}x^j + a_{id}x^i) =$
$= \left[\exp(a_{ij}x^i x^j)\right](a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + \left[\exp(a_{ij}x^i x^j)\right](a_{dj}\delta^j_c + a_{id}\delta^i_c) =$
$= \left[\exp(a_{ij}x^i x^j)\right]\left[(a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + (a_{dc} + a_{cd})\right]$
Resultando entonces

$$\partial_c\partial_d \exp(a_{ij}x^i x^j) = \left[\exp(a_{ij}x^i x^j)\right]\left[(a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + (a_{dc} + a_{cd})\right] \tag{2}$$

Respecto a $b$
$\partial_b\left[\exp(a_{ij}x^i x^j)\right]\left[(a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + (a_{dc} + a_{cd})\right] =$
$= \left[\exp(a_{ij}x^i x^j)\right](a_{bj}x^j + a_{ib}x^i) + \left[\exp(a_{ij}x^i x^j)\right]\partial_b[(a_{cj}x^j + a_{ic}x^i)(a_{dj}x^j + a_{id}x^i) + (a_{dc} + a_{cd})] =$
$= \left[\exp(a_{ij}x^i x^j)\right](a_{bj}x^j + a_{ib}x^i) + \left[\exp(a_{ij}x^i x^j)\right]\left\{\left[\partial_b(a_{cj}x^j + a_{ic}x^i)\right](a_{dj}x^j + a_{id}x^i) + (a_{cj}x^j + a_{ic}x^i)\partial_b(a_{dj}x^j + a_{id}x^i)\right\}$

$= \left[\exp(a_{ij}x^i x^j)\right]\left\{(a_{bj}x^j + a_{ib}x^i) + (a_{cj}\delta^j_b + a_{ic}\delta^i_b)(a_{dj}x^j + a_{id}x^i) + (a_{cj}x^j + a_{ic}x^i)(a_{dj}\delta^j_b + a_{id}\delta^i_b)\right\}$
$= \left[\exp(a_{ij}x^i x^j)\right]\left\{(a_{bj}x^j + a_{ib}x^i) + (a_{cb} + a_{bc})(a_{dj}x^j + a_{id}x^i) + (a_{cj}x^j + a_{ic}x^i)(a_{db} + a_{bd})\right\} =$
$= \left[\exp(a_{ij}x^i x^j)\right]\left\{(a_{bj}x^j + a_{ib}x^i) + 2x^i a_{bc}a_{di} + 2x^i a_{bd}a_{ci} + 2x^j a_{bc}a_{dj} + 2x^j a_{bd}a_{cj}\right\}$

Lo que nos da

$$\partial_b\partial_c\partial_d \exp(a_{ij}x^i x^j) = \left[\exp(a_{ij}x^i x^j)\right]\left\{(a_{ib} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})x^i + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})x^j\right\} \tag{3}$$

Por último derivando respecto a $a$
$\partial_a\left[\exp(a_{ij}x^i x^j)\right]\left\{(a_{bi} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})x^i + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})x^j\right\} =$
$= [\exp(a_{ij}x^i x^j)](a_{aj}x^j + a_{ia}x^i)\left\{(a_{bi} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})x^i + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})x^j\right\} +$
$+[\exp(a_{ij}x^i x^j)]\partial_a\left\{(a_{bi} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})x^i + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})x^j\right\} =$

Ahora ha llegado el momento de tener en cuenta que queremos el valor de esa cuarta derivada cuando $J = 0$ o sea $x = 0$ por lo que todo el primer término acabara siendo cero.
Nos quedamos así con solo el segundo término
$[\exp(a_{ij}x^i x^j)]\partial_a\left\{(a_{bi} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})x^i + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})x^j\right\} =$
$= [\exp(a_{ij}x^i x^j)]\left\{(a_{bi} + 2a_{bc}a_{di} + 2a_{bd}a_{ci})\delta^i_a + (a_{bj} + 2a_{bc}a_{dj} + 2a_{bd}a_{cj})\delta^j_a\right\} =$
$= [\exp(a_{ij}x^i x^j)]\left\{(a_{ba} + 2a_{bc}a_{da} + 2a_{bd}a_{ca}) + (a_{ba} + 2a_{bc}a_{da} + 2a_{bd}a_{ca})\right\}$

y por la simetría de las matrices $A$ , $A^{-1}$ y consecuentemente $a$

$$\partial_a\partial_b\partial_c\partial_d \exp(a_{ij}x^ix^j) = [\exp(a_{ij}x^ix^j)](2a_{ab} + 4a_{bc}a_{da} + 4a_{bd}a_{ca}) \tag{4}$$

y aplicando nuevamente $x = 0$
$\partial_a\partial_b\partial_c\partial_d \exp(a_{ij}x^ix^j) = (\frac{1}{m^2}a_{ab} + \frac{1}{m^4}(a_{bc}a_{da} + a_{bd}a_{ca})$
Revertiendo el cambio $a_{ij=}\frac{1}{2m^2}A^{-1}_{ij}$

$$\partial_a\partial_b\partial_c\partial_d \exp(a_{ij}x^ix^j) = \frac{1}{m^2}a_{ab} + \frac{1}{m^4}(a_{bc}a_{da} + a_{bd}a_{ca}) \tag{5}$$

Obteniendo finalmente

$$\langle\phi_a\phi_b\phi_c\phi_d\rangle = \frac{1}{m^2}A^{-1}_{ab} + \frac{1}{m^4}(A^{-1}_{ad}A^{-1}_{bc} + A^{-1}_{ac}A^{-1}_{bd}) \tag{6}$$

Dada la simetria de la expresión final es muy probable que el resultado sea correcto o casi correcto ...lo que nos pone...muy muy contentos :D

Ceuta, 1 de marzo de 2019
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